On a conjecture about the equation $A^{mx} + A^{my} = A^{mz}$

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Abstract. Let A be a given integral 2×2 matrix. We prove that the equation

$$(\star) \qquad A^{mx}+A^{my}=A^{mz}$$

has a solution in positive integers x,y,z and $m>2$ if and only if the matrix A is a nilpotent matrix or the matrix A has an eigenvalue $\alpha=\frac{1+i\sqrt{3}}{2}$.

1. Introduction

First we note that $(\star)$ is equivalent to the following Fermat's equation

$$(1) \qquad X^m + Y^m = Z^m, \quad m > 2,$$

where $X = A^x$, $Y = A^y$ and $Z = A^z$.

It has been recently proved by A. WILES [12], R. TAYLOR and A. WILES [11] that (1) has no solution in nonzero integers X, Y, Z if $m > 2$. But, in contrast to the classical case, the Fermat's equation (1) has infinitely many solutions in 2×2 integral matrices X, Y, Z for $m = 4$. This fact was discovered by R. Z. DOMIATY [2] in 1966. Namely, he proved that, if

$$X = \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix} \text{ and } Z = \begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix},$$

where a, b, c are integer solutions of the Pythagorean equation $a^2 + b^2 = c^2$, then

$$X^4 + Y^4 = Z^4.$$

Other results connected with Fermat's equation in the set of matrices are given in monograph [10] by P. RIBENBOIM. In these investigations it is an important problem to give a necessary and sufficient condition for the solvability of (1) in the set of matrices. Such type results were proved recently by A. KHAZANOV [7], when the matrices X, Y, Z belong to $SL_2(Z)$, $SL_3(Z)$ or $GL_3(Z)$. In particular, he proved that there are solutions of (1) in $X, Y, Z \in SL_2(Z)$ if and only if m is not a multiple of 3 or 4. We proved in [4] a necessary condition for the solvability of (1) in 2×2 integral matrices X, Y, Z having a determinant form. More precisely, we proved (see [4], Thm. 2) that the equation $(\star)$ does not hold in positive integers x, y, z and $m \geq 2$, if $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Another proof of this cited result was given by D. Frejman [3].

M. H. Le and Ch. Li [8] proved the following generalization of our result: Let $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ be a given integral matrix such that $r = \operatorname{Tr} A = a + d > 0$ and $\det A = ad - bc < 0$, then $(\star)$ does not hold.

In their paper they posed the following

Conjecture. Let A be an integral 2×2 matrix. The equation $(\star)$ has a solution in natural numbers x, y, z and $m > 2$ if and only if the matrix A is a nilpotent matrix.

A corrected version of this Conjecture was proved by the same authors in [9].

In the present paper we prove the following

Theorem. *The equation $(\star)$ has a solution in positive integers x, y, z and $m > 2$ if and only if the matrix A is a nilpotent matrix or the matrix A has an eigenvalue $\alpha = \frac{1+i\sqrt{3}}{2}$.*

We note that the condition matrix A has an eigenvalue $\alpha = \frac{1+i\sqrt{3}}{2}$ is equivalent to $\operatorname{Tr} A = \det A = 1$ (cf. [9]). On the other hand it is easy to see that the condition $\det A = 1$ implies that the matrix A cannot be a nilpotent matrix, thus the original Conjecture of M. H. Le and Ch. Li is not true.

We also note that X. Chen [1] proved that if A_n is the companion matrix for the polynomial $f(x) = x^n - x^{n-1} - \ldots - x - 1$ then the equation $(\star)$ with $A = A_n$ has no solution in positive integers x, y, z and $m \geq 2$ for any fixed integer $n \geq 2$.

Futher result of this type is contained by [5]. Namely, we proved the following:

Let $A = (a_{ij})_{n \times n}$ be a matrix with at least one real eigenvalue $\alpha > \sqrt{2}$. If the equation

$$A^r + A^s = A^t \tag{2}$$

has a solution in positive integers r, s and t then $\max\{r - t, s - t\} = -1$.

From this cited result one can obtain the corresponding results of the papers [1], [3], [4], [8] as particular cases.

2. Basic Lemmas

Lemma 1. *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be an integral matrix such that* $\operatorname{Tr} A \neq 0$ *or* $\det A \neq 0$ *and let*

$$r = a + d = \operatorname{Tr} A, \quad s = -\det A, \quad A_0 = r, \quad A_1 = rA_0 + s$$

and

$$A_n = rA_{n-1} + sA_{n-2} \quad \textit{if} \quad n \geq 2.$$

Then for every natural number $n \geq 2$, *we have*

$$A^n = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^n = \begin{pmatrix} aA_{n-2} + sA_{n-3} & bA_{n-2} \\ cA_{n-2} & dA_{n-2} + sA_{n-3} \end{pmatrix},$$

where we put $A_{-1} = 1$.

The proof of this Lemma immediately follows from Theorem 1 of [6].

Lemma 2. *Let A be an integral matrix satisfying the assumptions of Lemma 1 and let A_n be the recurrence sequence associated with the matrix A as in Lemma 1. Moreover, let Δ_n be the discriminant of the characteristic polynomial of A^n if $n \geq 2$ and let $\Delta_1 = \Delta = r^2 + 4s$. Then for every natural number $n \geq 2$ we have $\Delta_n = \Delta A_{n-2}^2$.*

The proof of Lemma 2 is given in [4].

Lemma 3. *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be an integral matrix and let* $f(x) = x^2 - (\operatorname{Tr} A)x + \det A$ *be the characteristic polynomial of A with the roots $\alpha, \beta \neq \frac{1+i\sqrt{3}}{2}$ and the discriminant $\Delta = r^2 + 4s$, where $r = a + d = \operatorname{Tr} A$ and $s = -\det A$. If $s \neq 0$ and $\Delta \neq 0$ then the equation $(\star)$ has no solutions in natural numbers x, y, z and $m > 2$.*

Proof. If $x = z$ and $(\star)$ is satisfied then $A^{my} = 0$, thus $\det A = 0$, which contradicts to our assumption. Similarly we obtain a contradiction when $y = z$. If $x = y$ then by $(\star)$ it follows that $2A^{mx} = A^{mz}$, hence $4(\det A)^{mx} = (\det A)^{mz}$ and so we obtain a contradiction, because the last equality is impossible in natural numbers x, y, z and $m > 2$ with integer $\det A \neq 0$.

Further on we can assume that if $(\star)$ is satisfied, then x, y and z are distinct natural numbers. Since $s = -\det A \neq 0$, therefore there exists the inverse matrix A^{-1} and from $(\star)$ we obtain

$$A^{m(x-z)} + A^{m(y-z)} = I, \quad \text{if} \quad \min\{x, y, z\} = z \tag{3}$$

$$A^{m(x-y)} + I = A^{m(z-y)}, \quad \text{if} \quad \min\{x, y, z\} = y, \tag{4}$$

$$I + A^{m(y-x)} = A^{m(z-x)}, \quad \text{if} \quad \min\{x, y, z\} = x, \tag{5}$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Let $\{A_n\}$ be the recurrence sequence associated with the matrix A. Then applying Lemma 1 to (3) we obtain

$$
(6)\quad \begin{aligned}
&a\left(A_{m(x-z)-2} + A_{m(y-z)-2}\right) - (\det A)\left(A_{m(x-z)-3} + A_{m(y-z)-3}\right) = 1,\\
&b\left(A_{m(x-z)-2} + A_{m(y-z)-2}\right) = 0,\\
&c\left(A_{m(x-z)-2} + A_{m(y-z)-2}\right) = 0,\\
&d\left(A_{m(x-z)-2} + A_{m(y-z)-2}\right) - (\det A)\left(A_{m(x-z)-3} + A_{m(y-z)-3}\right) = 1.
\end{aligned}
$$

From Lemma 1, (4) and (5) we obtain similar formulae to (6).

Suppose that $b \neq 0$ or $c \neq 0$. Then from (6) we get $\det A = \pm 1$. On the other hand since $\Delta \neq 0$, therefore from Lemma 2 we can deduce that

$$
(7)\qquad A_{n-2} = \frac{1}{\sqrt{\Delta}}\left(\alpha^n - \beta^n\right).
$$

Substituting (7) to (6) we obtain

$$
(8)\qquad \alpha^{m(x-z)} + \alpha^{m(y-z)} = \beta^{m(x-z)} + \beta^{m(y-z)} = 1.
$$

By (4) and (5) we similarly have

$$
(9)\qquad \alpha^{m(z-y)} - \alpha^{m(x-y)} = \beta^{m(z-y)} - \beta^{m(x-y)} = 1
$$

and

$$
(10)\qquad \alpha^{m(z-x)} - \alpha^{m(y-x)} = \beta^{m(z-x)} - \beta^{m(y-x)} = 1.
$$

From (8)–(10) it follows that in all cases

$$
(11)\qquad \alpha^{mx} + \alpha^{my} = \alpha^{mz} \quad \text{and} \quad \beta^{mx} + \beta^{my} = \beta^{mz}
$$

for natural numbers x, y, z and $m > 2$, which can be written in the forms

$$
(12)\qquad \alpha^{m(x-z)} + \alpha^{m(y-z)} = 1 \quad \text{and} \quad \beta^{m(x-z)} + \beta^{m(y-z)} = 1.
$$

Since $\Delta \neq 0$, thus we consider two cases: $\Delta > 0$ or $\Delta < 0$. Let us suppose that $\Delta > 0$. Since $\Delta = r^2 + 4s$ and $s = -\det A = \pm 1$, so we have $\Delta \geq 5$. If $r > 0$ then we obtain

$$
(13)\qquad \alpha = \frac{r + \sqrt{\Delta}}{2} \geq \frac{1 + \sqrt{5}}{2} > \sqrt{2} > 1.
$$

From (13) and (12) it follows that both exponents $m(x-z)$ and $m(y-z)$ must be negative. On the other hand fom (13) we have $\alpha^{-2} < \frac{1}{2}$ and by (12) it follows that it cannot happen that both exponents $m(x-z)$ and $m(y-z)$ are ≤ -2. Therefore one of them must be equal to -1 and we obtain $m(x-z) = -1$ or $m(y-z) = -1$. But this is impossible, because $m > 2$ and x, y, z are positive integers.

After this we consider the case $r \leq 0$. Let us suppose that $r < 0$ and put $r = -r'$, where $r' > 0$. Then we have

$$\beta = \frac{r-\sqrt{\Delta}}{2} = -\frac{r'+\sqrt{\Delta}}{2} = -\beta$$

and

$$\beta = r' + \sqrt{\frac{\Delta}{2}} \geq \frac{1+\sqrt{5}}{2} > \sqrt{2} > 1.$$

Substituting $\beta = -\beta$ to the second equation of (12) we obtain

$$(-1)^{m(x-z)}(\beta')^{m(x-z)} + (-1)^{m(y-z)}(\beta')^{m(y-z)} = 1. \tag{14}$$

If m is even then as in our previous case we obtain a contradiction. So, we can assume that m is an odd natural number greater than 2. If $x-z$ and $y-z$ are odd then it is easy to see that (14) does not hold. Therefore one of them must be even and from (14) we obtain

$$(\beta')^{m(x-z)} - (\beta')^{m(y-z)} = 1, \quad \text{if} \quad x-z \quad \text{is even and} \quad y-z \quad \text{is odd} \tag{15}$$

and

$$(\beta')^{m(y-z)} - (\beta')^{m(x-z)} = 1, \quad \text{if} \quad y-z \quad \text{is even and} \quad x-z \quad \text{is odd.} \tag{16}$$

Because of the symmetry, it is sufficient to consider one of these equations. Let us suppose that (15) is satisfied. If $x-z > 0$ and $y-z > 0$ then, by(15), it follows that $x-z > y-z$. On the other hand, (15) can be represented in the form

$$(\beta')^{m(y-z)}\left((\beta')^{m(x-z)} - 1\right) = 1. \tag{17}$$

The condition $x-z > y-z$ implies $x > y$ and since $\beta' > \sqrt{2}, m > 2, x-z > 0$ and $y-z > 0$, therefore (17) is impossible. Hence we get that one of the differences $x-z$ so $y-z$ must be negative. Suppose that $x-z < 0$ and $y-z > 0$. Then from (15)

$$(\beta')^{m(x-z)} = (\beta')^{m(y-z)} + 1 \tag{18}$$

follows. It is easy to see that $(\beta')^{m(x-z)} = \left((\beta')^{-2}\right)^{\frac{m(z-x)}{2}}$. On the other hand we have $(\beta')^{-2} < \frac{1}{2}$ and we obtain

$$(\beta')^{m(x-z)} = \left((\beta')^{-2}\right)^{\frac{m(z-x)}{2}} < \left(\frac{1}{2}\right)^{\frac{m(z-x)}{2}} < \frac{1}{2},$$

because $\frac{m(z-x)}{2} > 1$. Therefore from (18) we get

$$(\beta')^{m(y-z)} + 1 = (\beta')^{m(x-z)} < \frac{1}{2},$$

which is impossible. In a similar way we obtain a contradiction in the case $x-z > 0$ and $y-z < 0$. It remains to consider the case when both differences $x-z$ and $y-z$ are negative. From (15) we have

$$1 = \left|(\beta')^{m(x-z)} - (\beta')^{m(y-z)}\right| \leq (\beta')^{m(x-z)} + (\beta')^{m(y-z)}. \tag{19}$$

On the other hand we have

$$(\beta')^{m(x-z)} = \left((\beta')^{-2}\right)^{\frac{m(z-x)}{2}} < \left(\frac{1}{2}\right)^{\frac{m(z-x)}{2}} < \frac{1}{2} \tag{20}$$

and

$$(\beta')^{m(y-z)} + \left((\beta')^{-2}\right)^{\frac{m(z-y)}{2}} < \left(\frac{1}{2}\right)^{\frac{m(z-y)}{2}} < \frac{1}{2}. \tag{21}$$

Hence, by (19)–(21), we get a contradiction.

Further on we have to consider the case $r = 0$. But in this case we have $\alpha = 1, \beta = -1$ and we can can observe that (12) is impossible.

Now, we can consider the case $\Delta < 0$. Since $s = -\det A = \pm 1$ and $\Delta = r^2 + 4s < 0$, therefore we have $s = -1$ and the inequality $r^2 - 4 < 0$ implies $-2 < r < 2$, that is, $r = -1, 0, 1$.

The case $r = 1$ is impossble by the assumptions on the eigenvalues of the matrix A.

If $r = 0$ then we obtain that $\alpha = i, \beta = -i$ and it is easy to check that (12) does not hold.

If $r = -1$ then $\alpha = \frac{-1+i\sqrt{3}}{2}$ is the third root of unity. Analyzing the exponents $m(x-z)$ and $m(y-z)$ modulo 3 in (12) we get a contradiction.

Summarizing, we obtain that in the case $b \neq 0$ or $c \neq 0$ the equation $(\star)$ has no solution in positive integers x, y, z and $m > 2$. So, $b = c = 0$ and the matrix A can be reduced to a diagonal matrix of the form $A = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$. On the other hand for every natural number k we have

$$(22) \qquad A^k = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}^k = \begin{pmatrix} a^k & 0 \\ 0 & d^k \end{pmatrix}.$$

If $(\star)$ is satisfied then, by (22), it follfows that

$$(23) \qquad a^{mx} + a^{my} = a^{mz}, \quad d^{mx} + d^{my} = d^{mz}.$$

From the assumption of Lemma 3 we have $s = -\det A \neq 0$. This condition implies $ad \neq 0$, because $\det A = \det \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = ad$. Therefore (23) does not hold.

Considering all of the cases the proof of Lemma 3 is complete.

Now, we can prove the following.

Lemma 4. *Let* $A = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$ *be an integral matrix and let* $r = \operatorname{Tr} A, s = -\det A$ *and* $\Delta = r^2 + 4s$. *If* $s \neq 0$ *and* $\Delta = 0$, *then* $(\star)$ *has no solutions in positive integers* x, y, z *and* $m > 2$.

Proof. Since $s \neq 0$, therefore using Lemma 1 in similar way as in the proof of Lemma 3, for the case $b \neq 0$ or $c \neq 0$ we obtain $s = -\det A = \pm 1$. Since, $\Delta = r^2 + 4s = 0$, thus $s = -1$ and consequently $r^2 - 4 = 0$, so we have $r = \pm 2$. Therefore we get $\alpha = \beta = \frac{r}{2} = 1$ if $r = 2$ and $\alpha = \beta = -1$ if $r = -2$. From the well-known theorem of Schur it follows that for any given matrix A there is an unitary matrix P such that

$$(24) \qquad A = P^{\star} T P,$$

where T is the upper triangular matrix having on the main diagonal the eigenvalues of the matrix A.

Suppose that the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with integer entries has the eigenvalues α, β.

From (24) by easy induction we obtain

$$(25) \qquad A^k = P^{\star} T^k P$$

for every natural number k, where T^k is the upper triangular matrix with the eigenvalues α^k, β^k on the main diagonal. If $(\star)$ is satisfied then, by (25), it follows that

$$T^{mx} + T^{my} = T^{mz} \tag{26}$$

and from (26) we have

$$\alpha^{mx} + \alpha^{my} = \alpha^{mz}, \quad \beta^{mx} + \beta^{my} = \beta^{mz}. \tag{27}$$

Since in our case $\alpha = \beta = \pm 1$ so we can see that (27) does not hold. Therefore we have $b = c = 0$ and we get a contradiction as we have got it in the last step of the proof of Lemma 3. So the proof of Lemma 4 is complete.

Lemma 5. *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be an itegral matrix and let* $r = \operatorname{Tr} A, s = -\det A$ *and* $\Delta = r^2 + 4s$. *If* $s = 0$ *and* $\Delta \neq 0$ *then the equation* $(\star)$ *has no solution in positive integers* x, y, z *and* $m > 2$.

Proof. From the assumptions of Lemma 5 it follows that $r \neq 0$ and therefore we can use Lemma 1. Since $s = 0$ so, by Lemma 1, it follows that

$$A^k = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^k = \begin{pmatrix} ar^{k-1} & br^{k-1} \\ cr^{k-1} & dr^{k-1} \end{pmatrix} = r^{k-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = r^{k-1} A. \tag{28}$$

If $(\star)$ is satisfied then from (28) we obtain

$$r^{mx} + r^{my} = r^{mz}. \tag{29}$$

Being $r \neq 0$, it is easy to see that the equation (29) is impossible in positive integers x, y, z and $m > 2$. This proves Lemma 5.

3. Proof of the Theorem

Suppose that the equation $(\star)$ has a solution in postive integers x, y, z and $m > 2$. Then by Lemma 3, Lemma 4 and Lemma 5 it follows that $s = \det A = 0$ and $r = \operatorname{Tr} A = 0$ or the matrix A has an eigenvalue $\alpha = \frac{1+i\sqrt{3}}{2}$. In the case $s = r = 0$ we have $a = -d$ and $s = -\det A = -(ad - bc) = -\left(-d^2 - bc\right) = d^2 + bc = 0$ and also putting $d = -a$ we have $a^2 + bc = 0$. On the other hand we have

$$A^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{pmatrix} = \begin{pmatrix} a^2 + bc & br \\ cr & d^2 + bc \end{pmatrix}. \tag{30}$$

Substituting

$$r = 0, a^2 + bc = d^2 + bc = 0$$

to (30) we obtain that $A^2 = 0$, that is the matrix A is a nilpotent matrix with nilpotency index two.

Now, we suppose that the matrix A is nilpotent matrix, i.e. $A^k = 0$ for some natural number $k \geq 2$. Then it is easy to see that $(\star)$ is satisfied for all positive integers $x, y, z, m > 2$ such that $mx \geq k, my \geq k, mz \geq k$.

Suppose that the matrix A has an eigenvalue $\alpha = \frac{1+i\sqrt{3}}{2}$. Then it is easy to check that $\alpha^2 = \frac{-1+i\sqrt{3}}{2} = \varepsilon$ is a third root of unity. By an easy calculation we obtain

$$\alpha^n = \begin{cases} 1, & \text{if} \quad n = 6k, \\ -\varepsilon^2, & \text{if} \quad n = 6k+1, \\ \varepsilon, & \text{if} \quad n = 6k+2, \\ -1, & \text{if} \quad n = 6k+3, \\ \varepsilon^2, & \text{if} \quad n = 6k+4, \\ -\varepsilon, & \text{if} \quad n = 6k+5. \end{cases} \tag{31}$$

Applying (31) we obtain that $(\star)$ is satisfied if and only if the following relations are satisfied

$$mx \equiv r_1(\bmod\ 6), \quad my \equiv r_2(\bmod\ 6), \quad mz \equiv r_3(\bmod\ 6), \tag{32}$$

where

$$\langle r_1, r_2, r_3\rangle = \langle 0,2,1\rangle, \langle 0,4,5\rangle, \langle 1,3,2\rangle, \langle 1,5,0\rangle, \langle 2,4,3\rangle, \langle 2,0,1\rangle,$$

$$\langle 3,1,2\rangle, \langle 3,5,4\rangle, \langle 4,0,5\rangle, \langle 4,2,3\rangle, \langle 5,0,1\rangle, \langle 5,3,4\rangle.$$

The proof of Theorem is complete.

From the proof of Theorem we get the following

Corollary. *All soluitions of the equation$(\star)$ in natural numbers x, y, x and $m > 2$, when the matrix A has an eigennvalue $\alpha = \frac{1+i\sqrt{3}}{2}$ are given by the congruence formulas (32) with the above restrictions on $\langle r_1, r_2, r_3\rangle$ and if the matrix A is a nilpotent matrix with nilpotency index $k \geq 2$ then $(\star)$ is satisfied by all positive integers $x, y, z, m > 2$ such that $mx \geq k, my \geq k$ and $mz \geq k$.*

Remark. We note that Theorem with Corollary is equivalent to the result presented by M. H. Le and Ch. Li in [9], but our proof is given in another way and it gives more information about the impossibility of the solvability of $(\star)$ in the cases mentioned in Lemma 3, 4, 5.

References

[1] X. Chen, On Fermat's equation in the set of generalized Fibonacci matrices, *Discuss. Math., Algebra and Stochastic Methods* **17** (1997), 5–8.

[2] R. Z. Domiaty, Solutions of $x^4+y^4=z^4$ in 2×2 integral matrices, *Amer. Math. Monthly.* **73** (1966), 631.

[3] D. Frejman, On Fermat's equation in the set of Fibonacci matrices, *Discuss. Math.* **13** (1993), 61–64.

[4] A. Grytczuk, On Fermat's equation in the set of integral 2×2 matrices, *Period. Math. Hungar.* **30** (1995), 67–72.

[5] A. Grytczuk, Note on Fermat's type equation in the set of $n\times n$ matrices, *Discuss. Math., Algebra and Stochastic Methods*, **17** (1997), 19–23.

[6] A. Grytczuk and K. Grytczuk, *Functional recurrences Applications of Fibonacci Numbers,* Ed. G. E. Bergum et al., by Kluwer Acad. Publ., Dordrecht, 1990, 115–121.

[7] A. Khazanov, Fermat's equation in matrices, *Serdica Math. J.* **21** (1995), 19–40.

[8] M. H. Le and Ch. Li, A note on Fermat's equation in integral 2×2 matrices, *Discuss. Math., Algebra and Stochastic Methods,* **15** (1995), 135–136.

[9] M. H. Le and Ch. Li , On Fermat's equation in integral 2×2 matrices, *Period. Math. Hungar.* **31** (1995), 219–222.

[10] P. Ribenboim, *13 Lectures on Fermat's Last Theorem,* Springer Verlag, 1979.

[11] R. Taylor and A. Wiles, Ring-theoretic properties of certain Hecke algebras, *Annals of Math.* **141** (1995), 553–572.

[12] A. Wiles, Modular elliptic curves and Fermat's Last Theorem, *Annals of Math.* **141** (1995), 443–551.

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